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Refined analytical formulation for static bending and free vibrations of functionally graded beams

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Abstract. This paper presents a refined analytical theory for the bending and free vibration analysis of functionally graded (FG) beams accounting for transverse shear deformation. The material properties are assumed to vary continuously through the thickness according to a power-law distribution. Based on the adopted kinematic assumptions, the governing differential equations and effective stiffness relations are derived for both static and dynamic problems. Analytical solutions for the static bending of an FG beam under a uniformly distributed load and for the natural frequencies of free vibrations are obtained. A parametric study is performed to evaluate the influence of the material gradient index p and the beam slenderness ratio l/h. The results show that the dimensionless mid-span deflection increases from 6.2586 to 9.6986 when the gradient index changes from $p = 1$ to $p = 5$ for $l/h = 5$. The corresponding dimensionless natural frequency decreases from 3.9710 to 3.4025. Comparisons with the Timoshenko beam theory, Reddy's higher-order theory, and other refined models demonstrate very good agreement, with differences typically not exceeding 3-5%. The proposed formulation provides a realistic distribution of transverse shear stresses and converges to classical beam solutions for slender beams. The developed model can be effectively applied in the analysis and design of engineering structures made of functionally graded materials.
Keywords: functionally graded beam, shear deformation theory, static bending analysis, free vibration analysis, transverse shear stress, refined beam theory.

1. Introduction

Functionally graded materials (FGMs) constitute a class of composite materials characterized by a continuous variation of mechanical properties according to a prescribed distribution law, most commonly through the thickness of a structural element. In contrast to conventional laminated composites, which are characterized by distinct phase interfaces, FGMs provide a smooth variation of constituent volume fractions, thereby significantly reducing stress concentrations and eliminating the risk of delamination. Among various types of FGMs, metal-ceramic systems are the most widely employed, combining the high thermal resistance of the ceramic phase with the ductility and toughness of the metallic component [1].

The increasing implementation of functionally graded beams and plates in aerospace, energy, and civil engineering applications has created a strong demand for reliable theoretical models capable of accurately describing their static and dynamic behavior. Although exact solutions based on two-dimensional elasticity theory provide high accuracy, they are associated with substantial mathematical complexity and computational cost. Consequently, approximate beam theories have gained widespread application for practical analysis [2].

The classical Euler-Bernoulli beam theory (CBT) neglects transverse shear deformation and assumes that cross-sections remain plane and perpendicular to the neutral axis after deformation. As a result, this theory is generally applicable only to slender beams. To overcome this limitation, Timoshenko beam theory incorporates transverse shear deformation and allows independent rotation of the cross-sections. However, it assumes a constant shear strain distribution through the beam thickness and therefore requires the introduction of a shear correction factor to accurately predict transverse shear stresses [3]. To overcome these limitations, refined and higher-order shear deformation theories have been extensively developed in recent years.

A considerable amount of research has been devoted to the development of refined analytical and numerical models for the bending, buckling, and vibration analysis of functionally graded beams. The theoretical foundations of many higher-order structural models originate from the well-known higher-order shear deformation theory proposed by Reddy for laminated composite plates [4]. In this theory, a higher-order polynomial distribution of transverse shear strains through the thickness is assumed, which allows a more realistic representation of shear effects and automatically satisfies the zero transverse shear stress conditions on the top and bottom surfaces without the need for a shear correction factor. Later studies extended these concepts to structures with continuously varying material properties.

An exact elasticity solution for functionally graded beams was presented in [5], where the influence of material gradation on stress distribution and displacement fields was examined in detail. The free vibration characteristics of simply supported functionally graded (FG) beams were investigated in [6], demonstrating that the gradient index of material properties plays a significant role in determining the natural frequencies.

Further developments in the literature introduced refined beam formulations capable of accurately capturing shear deformation effects. A refined shear deformation theory suitable for the static and vibration analysis of FG beams was developed in [7], showing that reliable results can be obtained without employing shear correction factors. The applicability of this approach was later extended to sandwich configurations through a finite element formulation proposed in [8], which enabled the vibration and buckling behavior of functionally graded sandwich beams to be investigated.

Alternative numerical formulations were also proposed. For instance, a finite element model based on the first-order shear deformation theory was formulated in [9] to study the vibration and buckling responses of FG beams, highlighting the importance of shear deformation in structural analysis. An efficient deformation-based finite element approach for bending analysis was subsequently introduced in [10], providing accurate predictions with improved computational efficiency.

In parallel with these developments, several refined beam theories employing special forms of displacement fields have been proposed. A trigonometric shear deformation model introduced by Touratier in [11] provided an efficient representation of transverse shear deformation and became a basis for many subsequent refined theories. In this model, the transverse shear strain distribution through the thickness is described using a sinusoidal function, which allows a more realistic representation of shear effects compared with lower-order theories. An important advantage of this formulation is that the zero transverse shear stress condition on the top and bottom surfaces of the beam is satisfied automatically without requiring a shear correction factor. Later studies incorporated both shear and normal deformation effects in the modeling of FG beams. Such an approach was presented in [12], where a simplified theory accounting for transverse shear and normal strains was proposed.

Higher-order kinematic descriptions were further explored through the development of a fifth-order shear and normal deformation theory for power-law FG beams, which improved the prediction accuracy for bending and buckling responses [13]. Hyperbolic shear deformation functions were later employed for the analysis of FG beams subjected to bending, buckling, and free vibration [14]. A new higher-order shear deformation model suitable for functionally graded sandwich beams was proposed in [15], demonstrating good agreement with existing analytical and numerical solutions. More recently, refined hyperbolic shear deformation formulations have also been extended to the analysis of nonlocal functionally graded nanobeams [16].

Additional studies have focused on the development of Navier type analytical solutions for the static bending and free vibration analysis of functionally graded beams [17], as well as on the formulation of new refined finite element models [18]. In reference [19], Hadji et al. investigated the static and dynamic behavior of functionally graded beams using higher-order shear deformation theories under various boundary conditions. Their formulation accounts for the variation of material properties through the beam thickness and provides improved predictions of deflections, stresses, and natural frequencies compared with classical beam theories. The results demonstrated that higher-order shear deformation models are particularly effective for analyzing moderately thick functionally graded beams.

Despite the substantial body of published research, most existing approaches are primarily oriented either toward numerical implementation or toward specific boundary conditions. In addition, static and dynamic analyses are often formulated within different theoretical frameworks, which prevent the use of a unified mathematical basis and complicate the execution of consistent parametric studies.

In this context, the development of a unified refined analytical formulation is of particular relevance, enabling the investigation of both static bending and free vibration of functionally graded beams with continuously varying properties through the thickness within a single theoretical framework. Such an approach ensures consistency between stiffness and inertial characteristics and makes it possible to evaluate the influence of the material gradient law on deflections and natural frequencies.

In the present study, a refined analytical formulation is developed for the problem of static bending and free vibration of functionally graded beams. The material properties are assumed to vary through the thickness according to a power-law distribution. The governing differential equations are derived, analytical expressions for deflections and natural frequencies are obtained, and a parametric analysis is performed to assess the influence of the material gradient index and the geometric characteristics of the beam on its structural response.

2. Methods

A functionally graded beam of length l with a rectangular cross-section $b \times h$ is considered, where b denotes the width and h the thickness of the beam. The geometry of the structure and the adopted coordinate system are shown in Figure 1. The coordinate axes x_1 and x_3 are oriented along the longitudinal and thickness directions of the beam, respectively. The beam is considered in the domain: $0 \leq x_1 \leq l;\ -\frac{h}{2} \leq x_3 \leq \frac{h}{2}$.

Figure 1 – Computational model of the functionally graded beam and the adopted coordinate system

The cross-section of the beam is assumed to remain constant along its length and to retain a rectangular shape. For the purpose of simplifying the subsequent mathematical derivations, the beam width may be taken as unity (b = 1), which does not reduce the generality of the obtained results.

It is assumed that the mechanical properties of the material vary continuously through the thickness in the x_3 direction according to a power-law distribution, corresponding to the class of power-law functionally graded materials (P-FGM) [20]:

$$\mathrm{E}(x_3) = E_m + (E_c - E_m)\left(\frac{1}{2} + \frac{x_3}{h}\right)^{\mathrm{p}}, \quad \rho(x_3) = \rho_m + (\rho_c - \rho_m)\left(\frac{1}{2} + \frac{x_3}{h}\right)^{\mathrm{p}}, \quad \mathrm{p} \geq 0, \tag{1}$$

where: p – denotes the material gradient index; E_m and E_c – represent the Young's moduli of the metallic and ceramic constituents; ρ_m and ρ_c – correspond to the mass densities of the metallic and ceramic phases, respectively.

The shear modulus is defined as:

$$G(x_3) = \frac{\mathrm{E}(x_3)}{2(1+\nu)} \tag{2}$$

Within the framework of the present refined shear deformation theory, the displacement field is assumed in a general form based on the kinematic assumptions of higher-order beam theories [21]:

$$U_1(x_1, x_3) = \phi(x_3)\theta(x_1), \quad \theta(x_1) = -\frac{dW_0}{dx_1} \tag{3}$$

$$U_3(x_1, x_3) = f(x_3)W(x_1), \quad W(x_1) = W_0(x_1) - \frac{D_{FG}}{A_{FG}}\frac{d^2W_0}{dx_1^2} \tag{4}$$

The thickness-wise distribution functions are expressed as follows [21]:

$$\phi(x_3) = \mathrm{h}\left(\frac{3}{2}\frac{x_3}{\mathrm{h}} - 2\frac{x_3^3}{\mathrm{h}^3}\right), \; f(x_3) = \frac{3}{2} - 6\frac{x_3^2}{\mathrm{h}^2} \tag{5}$$

For the selected functions, the following condition is satisfied, which ensures a physically consistent distribution of transverse shear strains across the thickness.

$$\frac{d\phi(x_3)}{dx_3} = f(x_3) \tag{6}$$

The function $\varphi(x_3)$ describes the distribution of axial displacements through the beam thickness, while the function $f(x_3)$ governs the distribution of transverse shear deformation. In the proposed model these functions are related by Eq. (6), which ensures the kinematic consistency of the formulation. Moreover, the adopted form of $f(x_3)$ automatically satisfies the zero transverse shear stress condition at the top and bottom surfaces of the beam.

The longitudinal strain is defined as follows:

$$\varepsilon_1 = \frac{\partial U_1}{\partial x_1} = \phi(x_3)\frac{d\theta}{dx_1} = -\phi(x_3)\frac{d^2W_0}{dx_1^2} \tag{7}$$

The transverse shear strain, taking into account the adopted assumptions, is expressed as follows [3]:

$$\gamma_{13} = \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} \tag{8}$$

Taking into account relations (3) and (4), we obtain:

$$\gamma_{13} = -f(x_3)\frac{D_{FG}}{A_{FG}}\frac{d^3W_0}{dx_1^3} \tag{9}$$

The stress components are determined according to Hooke's law:

$$\sigma_1 = E(x_3)\varepsilon_1, \quad \tau_{13} = G(x_3)\gamma_{13} \tag{10}$$

Substituting expressions (7) and (9), we obtain:

$$\sigma_1(x_1, x_3) = -E(x_3)\phi(x_3)\frac{d^2W_0}{dx_1^2}, \quad \tau_{13}(x_1, x_3) = -G(x_3)f(x_3)\frac{D_{FG}}{A_{FG}}\frac{d^3W_0}{dx_1^3} \tag{11}$$

The bending moment, taking into account relation Eq. (11), is determined as follows:

$$M = -\frac{d^2W_0}{dx_1^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(x_3)\phi(x_3)x_3dx_3$$

The effective bending stiffness of the FG beam is defined as:

$$D_{FG} = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(x_3)\phi(x_3)x_3dx_3 \tag{12}$$

Then:

$$M = -D_{FG}\frac{d^2W_0}{dx_1^2} \tag{13}$$

The transverse shear force, after substituting relation Eq. (11), is expressed as:

$$Q = -\frac{D_{FG}}{A_{FG}}\frac{d^3W_0}{dx_1^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} G(x_3)f(x_3)dx_3$$

The effective shear stiffness is introduced as follows:

$$A_{FG} = \int_{-\frac{h}{2}}^{\frac{h}{2}} G(x_3)f^2(x)dx_3 \tag{14}$$

Then:

$$Q = -D_{FG}\frac{d^3W_0}{dx_1^3} \tag{15}$$

The governing differential equation for the static bending of the functionally graded beam takes the form:

$$D_{FG}\frac{d^4W_0}{dx_1^4} = q(x_1) \tag{16}$$

Free vibrations are governed by the equation:

$$D_{FG}\frac{d^4W_0}{dx_1^4} - m_{FG}\omega^2W_0 = 0 \tag{17}$$

$$m_{FG} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(x_3)dx_3 \tag{18}$$

To verify the validity of the developed formulation and to assess its physical capabilities, a numerical analysis is carried out. The numerical analysis was performed using the derived analytical expressions implemented in Mathcad 15.0 (Parametric Technology Corporation, Needham, MA, USA) [22]. Problems of static bending and free vibration of functionally graded beams are considered, and the obtained results are compared with those predicted by established refined beam theories. Such a comparison makes it possible to confirm the accuracy and stability of the proposed model.

In the present study, the bending and free vibration behavior of FG beams with a rectangular cross-section is investigated. The beam has length l, thickness h, and unit width (b = 1). The geometry and coordinate system are defined such that the longitudinal axis coincides with the x_1 coordinate, while the x_3 axis is directed through the thickness of the beam. The beam is assumed to be simply supported at both ends (S-S boundary conditions).

The beam is composed of a metal-ceramic functionally graded material: Ceramic phase: alumina (E_c = 380 GPa, ν = 0.3), metallic phase: aluminum (E_m = 70 GPa, ν = 0.3), the mass density of the metal is ρ_m = 2700 kg/m^3, and the mass density of the ceramic is ρ_c = 3960 kg/m^3.

For convenience, the following dimensionless forms are introduced:

– Transverse displacement: $\overline{W} = 100\frac{\mathrm{E_m}h^3}{\mathrm{q_0}l^4}W\left(\frac{l}{2},0\right)$;

– Axial stress: $\overline{\sigma_1} = \frac{\mathrm{h}}{\mathrm{q_0}l}\sigma_1\left(\frac{l}{2},\frac{h}{2}\right)$;

– Transverse shear stress: $\overline{\tau_{13}} = \frac{\mathrm{h}}{\mathrm{q_0}l}\ \tau_{13}\ (0,0)$;

– Dimensionless natural frequency: $\overline{\omega} = \frac{\omega l^2}{h}\sqrt{\frac{\rho_m}{E_m}}$.

3. Results and Discussion

Using the developed analytical model, numerical results are obtained for the analysis of bending and free vibration of functionally graded beams. Particular attention is devoted to examining the influence of the material gradient index, beam slenderness ratio, and transverse shear effects on the stress-strain state of FG beams. To evaluate the performance of the proposed formulation, the calculated deflections, stress distributions, and free vibration characteristics are presented in Tables 1-2 and Figures 2-6.

Table 1 – Comparison of dimensionless deflections and stresses of functionally graded beams subjected to uniformly distributed loads for various values of the power-law gradient index

p	Theory	$l/h = 5$			$l/h = 20$		
		$\overline{W}$	$\overline{\sigma_1}$	$\overline{\tau_{13}}$	$\overline{W}$	$\overline{\sigma_1}$	$\overline{\tau_{13}}$
1	Present	6.2586	5.8077	0.7187	5.7292	22.9038	0.7259
	Hadji et al.	6.1805	6.0709	0.7883	5.6965	24.0095	0.7890
	Reddy	6.2594	5.8836	0.7330	5.5685	23.2051	0.7432
	Touratier	6.2586	5.8892	0.7549	5.8049	23.2067	0.7686
	CBT	5.7746	5.7959	-	5.7746	23.1834	-
5	Present	9.6986	8.0059	0.5786	8.7031	31.3997	0.5863
	Hadji et al.	9.6933	8.3581	0.6523	8.6182	32.8183	0.6540
	Reddy	9.8281	8.1104	0.5904	8.8182	31.8127	0.6013
	Touratier	9.8367	8.1222	0.6155	8.8188	31.8159	0.6292
	CBT	8.7508	7.9428	-	8.7508	31.7711	-

Table 2 – Dimensionless numerical results for the free vibration of a simply supported (S-S) FG beam: comparison of the present model with Timoshenko and Reddy-type theories

p	Theory	$l/h = 5$	$l/h = 20$
		$\overline{\omega}$	$\overline{\omega}$
1	Present (HOBT)	3.9710	4.2038
	Present (FOBT)	3.9710	4.2038
	Timoshenko	3.9902	4.2050
	Reddy	3.9904	4.2050
5	Present (HOBT)	3.3740	3.6460
	Present (FOBT)	3.4025	3.6490
	Timoshenko	3.4312	3.6508
	Reddy	3.4012	3.6484

In the present study, the functionally graded beam is assumed to be composed of a metal and a ceramic phase. Following commonly used FG material models in the literature, aluminum is considered as the metal phase and alumina (Al_2O_3) as the ceramic phase, and Poisson's ratio is assumed to be constant through the thickness with $\nu = 0.3$.

For the purpose of comparison with existing beam theories, the deflection, normal stress, transverse shear stress, and natural frequency are presented in dimensionless form. Therefore, the normalized quantities $\overline{W}$, $\overline{\sigma_1}$, $\overline{\tau_{13}}$ and $\overline{\omega}$ are used to eliminate the influence of geometric parameters and loading conditions and to ensure a consistent comparison between different theories.

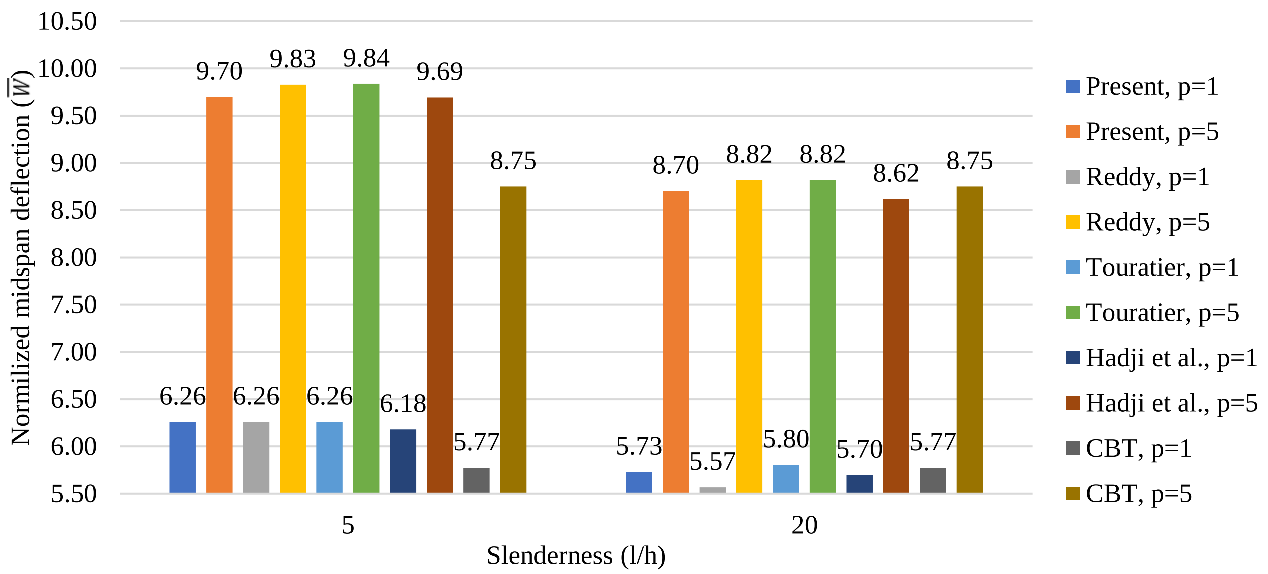

Figure 2 – Comparison of the normalized mid-span deflection obtained by different beam theories

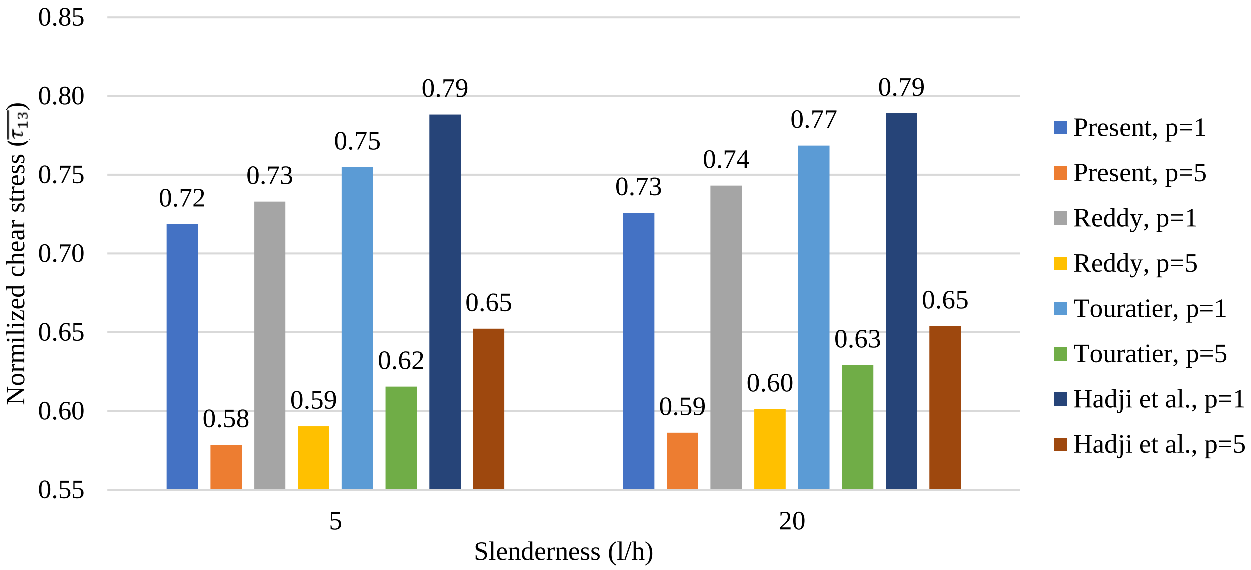

Figure 3 – Comparison of the convergence of transverse shear stresses obtained using refined theories for functionally graded beams

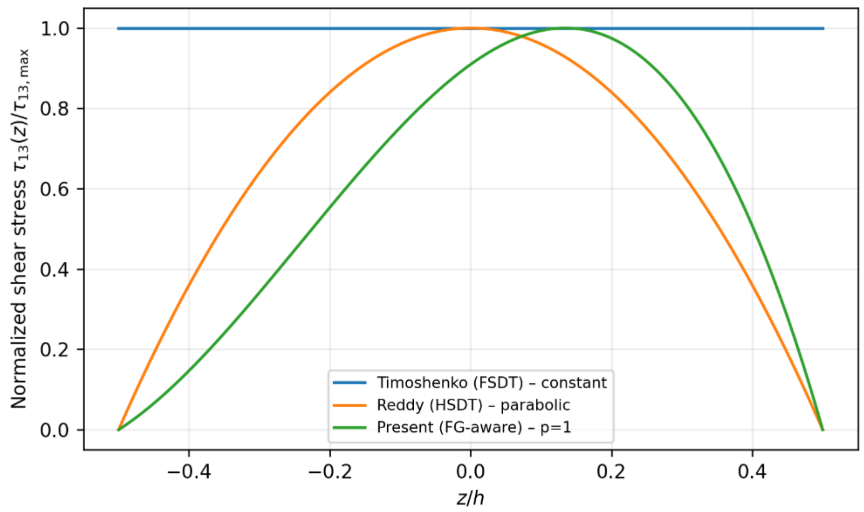

Figure 4 – Through-thickness distribution of transverse shear stresses (p = 1)

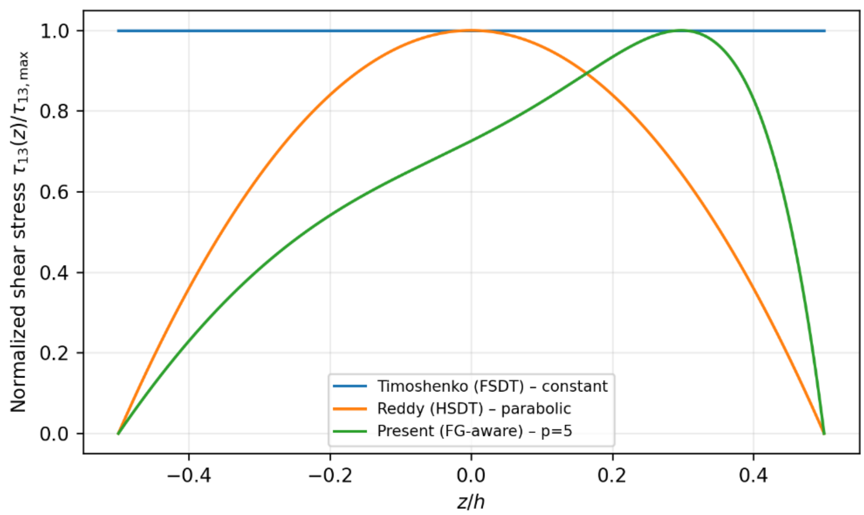

Figure 5 – Through-thickness distribution of transverse shear stresses (p = 5)

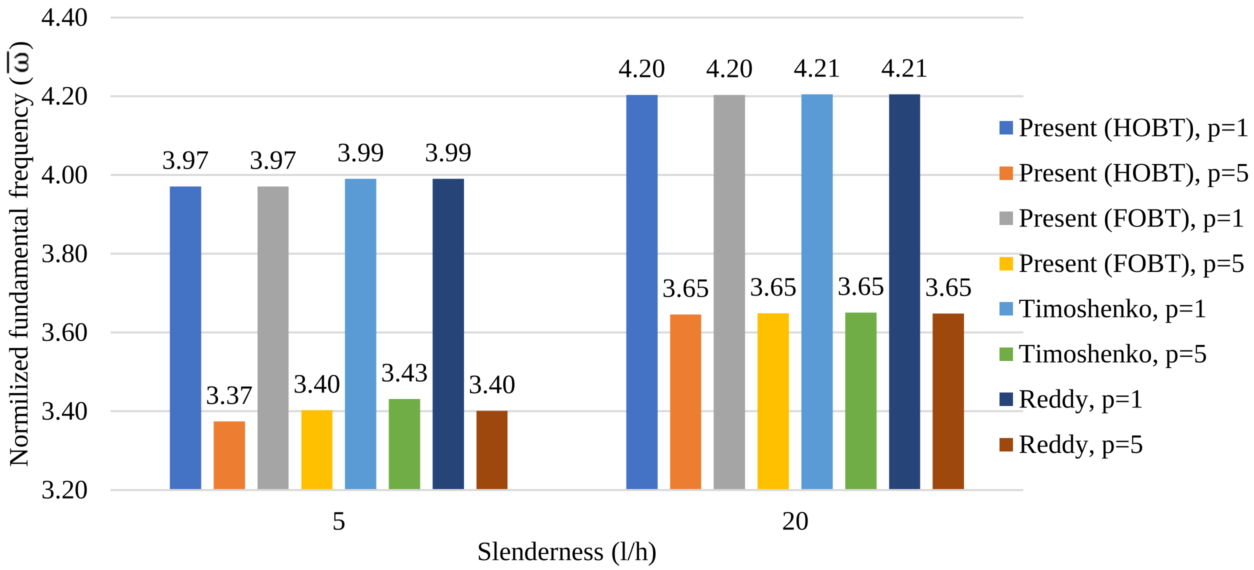

Figure 6 – Free vibration (S-S) comparison of the fundamental frequency: HOBT – Higher order beam theory; FOBT – First order beam theory.

The numerical results presented in Tables 1-2 and Figures 2-6 demonstrate that the proposed formulation provides stable predictions for both the static and dynamic behavior of functionally graded beams. The obtained results show good agreement with well-established refined beam theories reported in the literature, including the models of Timoshenko [3], Reddy [4], and Touratier [11]. This agreement confirms that the adopted kinematic assumptions and the derived governing equations are physically reasonable and mathematically consistent.

The parameter p used in Figures 2-6 represents the material gradation index, which describes the variation of material properties through the thickness of the functionally graded beam according to a power-law distribution. In particular, the parameter p controls the relative volume fractions of the metal and ceramic phases in the material. When $p = 0$, the material corresponds to a fully ceramic beam, whereas increasing values of p lead to a gradual increase in the metallic phase. The value $p = 1$ corresponds to an approximately linear variation of material properties, while $p = 5$ represents a more pronounced material gradation.

The bending analysis indicates that the main parameters of the stress-strain state, such as transverse deflection, normal stresses, and transverse shear stresses, are strongly affected by the beam slenderness ratio l/h. According to the values presented in Table 1, the discrepancies between the considered beam theories are more noticeable for relatively thick beams with $l/h = 5$. This behavior can be explained by the significant influence of transverse shear deformation in moderately thick structures. Similar conclusions were reported in earlier studies on functionally graded beams, where refined shear deformation theories were shown to provide more accurate predictions for thick beams [2], [4].

Figures 2 and 3 illustrate the influence of the slenderness ratio on the bending response of the beam. As the ratio l/h increases, the predictions of different beam theories gradually converge. This tendency corresponds to the classical asymptotic transition toward the Euler-Bernoulli beam theory, which becomes accurate for slender beams where transverse shear effects are negligible. A similar convergence behavior has also been reported in previous analytical studies of FG beams [2].

The distribution of transverse shear stresses through the beam thickness is presented in Figures 4-5. In contrast to classical beam models, where the shear stress profile depends only on geometric shape functions, the proposed formulation explicitly accounts for the variation of the material properties through the thickness. As a result, the transverse shear stress τ_{13} becomes dependent on the material gradation. When the gradient index p increases, the stress distribution becomes asymmetric and the location of the maximum stress shifts toward the stiffer region of the cross-section. Similar asymmetric stress distributions were reported in elasticity-based solutions for functionally graded beams [6].

The results of the free vibration analysis presented in Table 2 and Figure 6 confirm the correctness of the dynamic formulation. The obtained dimensionless natural frequencies show good agreement with those predicted by refined beam theories, including the Timoshenko model [3] and higher-order theories such as the formulation proposed by Reddy [4]. The differences between the models are more noticeable for thick beams, while for slender beams (l/h = 20), the results practically coincide. This behavior indicates that the proposed model correctly reproduces the transition to the classical thin-beam solution.

The influence of the material gradient index p is clearly observed in both static and dynamic responses of the beam. An increase in the value of p leads to a reduction in the effective bending stiffness and modifies the mass distribution through the thickness. Consequently, larger deflections and lower natural frequencies are obtained. This trend is consistent with the physical characteristics of functionally graded materials and agrees with the conclusions of previous investigations of FG beam behavior [2], [19].

Overall, the comparison of tabulated data and graphical results demonstrates that the proposed theory provides an energetically consistent description of the behavior of functionally graded beams. The model adequately accounts for transverse shear deformation effects, thickness-wise variation of mechanical properties, and geometric parameters without resorting to empirical correction factors. As a result, it achieves a balance between the computational efficiency of one-dimensional beam models and a level of physical fidelity approaching that of three-dimensional elasticity solutions, making the proposed approach a promising tool for the engineering analysis of advanced functionally graded structures.

From a practical engineering perspective, the obtained results can be useful for the analysis and design of structural elements made of functionally graded materials in civil engineering applications. Functionally graded beams may be used in modern building structures, bridge elements, layered structural members, and advanced composite components where gradual variation of material properties improves structural performance. The presented analytical formulation allows engineers to estimate deflections, stresses, and natural frequencies of such elements with sufficient accuracy. This information is important for evaluating structural stiffness, strength, and vibration resistance of load-bearing components in building structures.

4. Conclusions

In the present study, a refined analytical formulation has been developed for the bending and free vibration analysis of functionally graded beams, taking into account transverse shear deformation effects. The material properties are assumed to vary continuously through the beam thickness according to a power-law distribution, allowing a realistic representation of the inhomogeneous structure of FG materials.

The main findings of the study can be summarized as follows:

1. A unified analytical formulation has been established that allows the description of both static bending and free vibration of functionally graded beams within a consistent mechanical framework.

2. The obtained numerical results demonstrate good agreement with well-known refined beam theories, including the models of Timoshenko, Reddy, etc., confirming the accuracy and reliability of the proposed formulation.

3. The results show correct asymptotic convergence with increasing beam slenderness ratio l/h, indicating the mathematical consistency of the model and its compatibility with the classical thin-beam theory.

4. The developed formulation provides an explicit analytical representation of the transverse shear stresses $\tau_{13}(x_3)$, which depend on the thickness-wise variation of the shear modulus $G(x_3)$.

This feature allows the model to capture asymmetric stress distributions typical for functionally graded materials.

5. The influence of the material gradient index p on both static and dynamic responses has been demonstrated. Increasing the value of p leads to a redistribution of stiffness and mass along the beam thickness, which results in larger deflections and lower natural frequencies.

6. The dynamic behavior of the FG beam is governed by the relationship $\omega_n \sim \sqrt{\frac{D_{FG}}{m_{FG}}}$, emphasizing the importance of simultaneously accounting for the spatial variation of both stiffness and inertial properties.

Overall, the proposed formulation represents an efficient analytical tool for the study of functionally graded beams. Due to its relatively simple analytical structure and physically consistent predictions, the model can be applied to the analysis and design of advanced structural elements made of functionally graded and composite materials. The formulation can also be extended to problems involving elastic foundations, stability analysis, and structural dynamics of more complex engineering systems.

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References

[1] V. Birman and L. W. Byrd, “Modeling and Analysis of Functionally Graded Materials and Structures,” *Applied Mechanics Reviews*, vol. 60, no. 5, pp. 195–216, Sep. 2007, doi: 10.1115/1.2777164.

[2] H.-T. Thai and T. P. Vo, “Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories,” *International Journal of Mechanical Sciences*, vol. 62, no. 1, pp. 57–66, Sep. 2012, doi: 10.1016/j.ijmecsci.2012.05.014.

[3] S. P. Timoshenko, “On the correction for shear of the differential equation for transverse vibrations of prismatic bars,” *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 41, no. 245, pp. 744–746, May 1921, doi: 10.1080/14786442108636264.

[4] J. N. Reddy, “A Simple Higher-Order Theory for Laminated Composite Plates,” *Journal of Applied Mechanics*, vol. 51, no. 4, pp. 745–752, Dec. 1984, doi: 10.1115/1.3167719.

[5] M. Aydogdu and V. Taskin, “Free vibration analysis of functionally graded beams with simply supported edges,” *Materials & Design*, vol. 28, no. 5, pp. 1651–1656, Jan. 2007, doi: 10.1016/j.matdes.2006.02.007.

[6] B. V. Sankar, “An elasticity solution for functionally graded beams,” *Composites Science and Technology*, vol. 61, no. 5, pp. 689–696, Apr. 2001, doi: 10.1016/S0266-3538(01)00007-0.

[7] T. P. Vo, H.-T. Thai, T.-K. Nguyen, and F. Inam, “Static and vibration analysis of functionally graded beams using refined shear deformation theory,” *Meccanica*, vol. 49, no. 1, pp. 155–168, Jan. 2014, doi: 10.1007/s11012-013-9780-1.

[8] T. P. Vo, H.-T. Thai, T.-K. Nguyen, A. Maheri, and J. Lee, “Finite element model for vibration and buckling of functionally graded sandwich beams based on a refined shear deformation theory,” *Engineering Structures*, vol. 64, pp. 12–22, Apr. 2014, doi: 10.1016/j.engstruct.2014.01.029.

[9] V. Kahya and M. Turan, “Finite element model for vibration and buckling of functionally graded beams based on the first-order shear deformation theory,” *Composites Part B: Engineering*, vol. 109, pp. 108–115, Jan. 2017, doi: 10.1016/j.compositesb.2016.10.039.

[10] H. Ziou, M. Himeur, H. Guenfoud, and M. Guenfoud, “An Efficient Finite Element Formulation Based on Deformation Approach for Bending of Functionally Graded Beams,” *JSM*, vol. 12, no. 2, 2020, doi: 10.22034/jsm.2019.1867884.1437.

[11] M. Touratier, “An efficient standard plate theory,” *International Journal of Engineering Science*, vol. 29, no. 8, pp. 901–916, Jan. 1991, doi: 10.1016/0020-7225(91)90165-Y.

[12] M. Bourada, K. Abdelhakim, M. S. A. Houari, and A. Tounsi, “A new simple shear and normal deformations theory for functionally graded beams,” *Steel and Composite Structures*, vol. 18, pp. 409–423, Feb. 2015, doi: 10.12989/scs.2015.18.2.409.

[13] S. M. Ghumare and A. S. Sayyad, “A New Fifth-Order Shear and Normal Deformation Theory for Static Bending and Elastic Buckling of P-FGM Beams,” *Lat. Am. j. solids struct.*, vol. 14, no. 11, pp. 1893–1911, 2017, doi: 10.1590/1679-78253972.

[14] A. Sayyad and Y. Ghugal, "Bending, buckling and free vibration responses of hyperbolic shear deformable FGM beams," *MACS*, vol. 5, no. 1, Apr. 2018, doi: 10.22075/macs.2018.12214.1117.

[15] T.-K. Nguyen, T. Truong-Phong Nguyen, T. P. Vo, and H.-T. Thai, "Vibration and buckling analysis of functionally graded sandwich beams by a new higher-order shear deformation theory," *Composites Part B: Engineering*, vol. 76, pp. 273–285, Jul. 2015, doi: 10.1016/j.compositesb.2015.02.032.

[16] D. Indronil and I. M. Nazmul, "A novel hyperbolic shear deformation beam theory for functionally graded nonlocal nanobeams," *Discov Appl Sci*, vol. 7, no. 10, p. 1204, Oct. 2025, doi: 10.1007/s42452-025-07047-5.

[17] Y. L. Pei and L. X. Li, "Comment on the Navier's solution in 'A sinusoidal beam theory for functionally graded sandwich curved beams' (Composite Structures 226 (2019) 111246)," *Composite Structures*, vol. 243, p. 112248, Jul. 2020, doi: 10.1016/j.compstruct.2020.112248.

[18] F. Rahmani, R. Kamgar, and R. Rahgozar, "Finite Element Analysis of Functionally Graded Beams using Different Beam Theories," *Civ Eng J*, vol. 6, no. 11, pp. 2086–2102, Nov. 2020, doi: 10.28991/cej-2020-03091604.

[19] L. Hadji, N. Zouatnia, and F. Bernard, "An analytical solution for bending and free vibration responses of functionally graded beams with porosities: Effect of the micromechanical models," *Structural Engineering & Mechanics*, vol. 69, no. 2, pp. 231–241, Feb. 2019, doi: 10.12989/sem.2019.69.2.231.

[20] A. S. Sayyad and Y. M. Ghugal, "Analytical solutions for bending, buckling, and vibration analyses of exponential functionally graded higher order beams," *Asian J Civ Eng*, vol. 19, no. 5, pp. 607–623, Jul. 2018, doi: 10.1007/s42107-018-0046-z.

[21] K. A. Tursynov, "Arqalyqty esepteu teoriyalary," in *Textbook*, Karaganda: Karaganda Buketov University, 2007.

[22] B. Maxfield, "Essential PTC® Mathcad Prime® 3.0: a guide for new and current users," Oxford: Academic Press, 2014, p. 563.

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