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Article

Refined methodology for the analysis of beams on elastic foundations

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Abstract. The study proposes a refined method for analyzing beams on a two-parameter elastic foundation, overcoming the limitations of the classical Winkler model. Unlike the traditional approach, which considers soil deformation only in the applied load zone, the proposed methods introduce an additional parameter of bending stiffness, providing a more accurate description of beam-foundation interaction. A governing differential equation was derived, and its analytical solutions are presented for various boundary conditions and loading types. The numerical analysis results show that the distribution of vertical displacement, bending moment, and shear force along the normalized length of the beam is symmetric with respect to the midspan. It has been established that the maximum values of vertical displacement and bending moment are observed at the midspan: the vertical displacement reaches 0.000999157, while the bending moment attains 0.124892. At the same time, the shear force reaches its maximum value near the beam supports, amounting to 0.49966. The results indicate that the stress-strain state critical points of the beam on an elastic foundation are concentrated at the midspan (for displacement and bending moment) and at the supports (for shear force). The analysis demonstrates that the maximum shear stresses occur near the fixed end of the beam ($x = 0, z = 0$), gradually decrease to zero at midspan, and reach negative values at the opposite end ($x = 1, z = 0$). The normal stresses vary linearly along the cross-sectional height, from negative in the lower zone ($x=1/2, z = -1/2$) to positive in the upper zone ($x=1/2, z = 1/2$), with values close to zero near the neutral axis ($x=1//2, z = 0$). Comparison with the classical Winkler model shows close agreement in displacements, bending moments, and shear forces, while the proposed method provides improved accuracy in predicting normal and shear stress distributions.

Keywords: beam, two-parameter elastic foundation, Winkler model, displacement, bending moment, shear force, normal stress, shear stress.

1. Introduction

The analysis of the stress-strain state (SSS) of structural elements interacting with elastic foundations remains a fundamental and practically significant problem in structural mechanics. Beams on elastic foundations are widely used in civil engineering, transportation infrastructure, mechanical engineering, and related fields, where the effect of the supporting medium on structural performance must be incorporated into design models. Accurate evaluation of internal stresses, particularly normal and shear components, is crucial for ensuring strength and serviceability of structural elements, thereby improving the reliability of engineered systems. However, classical models tend to oversimplify the representation of the elastic foundation, which significantly reduces the accuracy and reliability of practical analysis [1].

The object of this study is a beam resting on a two-parameter elastic foundation and subjected to distributed loading, a structural system widely used in civil and transportation infrastructure. This issue becomes especially important under distributed loading, typical of real operating conditions, where complex stress distributions may arise in beams with variable stiffness, multilayer configurations, or those supported by elastic foundations.

Classical analytical approaches, such as Winkler's foundation model, assume a homogeneous medium and disregard shear stresses, which limits their applicability for realistic SSS assessment. However, distributed loading may generate complex stress distributions, especially in beams with variable stiffness, multilayer configurations, or those supported by heterogeneous foundations. Contemporary design practice, therefore, requires advanced models that account for both normal and shear stresses, interlayer interactions, and variable foundation properties. In this context, many recent contributions by different researchers have introduced refined formulations and numerical approaches to enhance the accuracy of SSS analysis and extend the applicability of theoretical models to practical engineering structures. The use of semi-analytical and numerical analytical methods, including the Ritz-Timoshenko approach and the contact layer method, enhances the accuracy of stress analysis and allows for adaptation to real engineering conditions [2]. Therefore, the topic of the present study is both timely and of considerable practical significance.

The theoretical background of this problem was established in the classical works of Winkler, Inglis, Bolotin, Timoshenko, and Muravskii, where linear models of elastic foundations and simplified loading schemes were introduced. Winkler's classical model, which assumes a linear relationship between load and settlement, has been extensively applied in engineering practice; however, the foundation is generally considered homogeneous [3], [4]. Fundamental contributions to the theory of beams on elastic foundations have addressed distributed loading, stability, and vibration problems, while also refining beam models to account for shear deformation. These developments significantly expanded the applicability of classical approaches to more realistic engineering conditions [5].

Subsequent studies by Levontev, Vlasov, and other researchers [6], [7] proposed refined approaches that incorporate the variability of the subgrade reaction coefficient along the length of the structure, thereby providing a more adequate representation of real foundation behavior.

The advancement of scientific thought has led to the development of more sophisticated models that incorporate variable stiffness characteristics, dissipative properties of the foundation, and the effects of non-stationary or moving loads. Among the effective numerical approaches for modeling the stress-strain state of beams on elastic foundations with variable parameters is the nodal acceleration method [8]. This method combines high computational accuracy with efficiency in resource utilization. The current stage of research is characterized by a transition from simplified analytical schemes to highly detailed numerical-analytical models, which enable the solution of applied problems in transportation and civil infrastructure.

In engineering practice, composite beams with intermediate hinges are frequently encountered, and the compliance of these hinges has a significant effect on the stress-strain state of the structure. The problem of accounting for elastic intermediate hinges has long remained insufficiently investigated, despite its practical relevance for designing structures under variable stiffness and consolidation conditions. An improved mathematical model was proposed in [9], [10] based on the introduction of corrective terms into the differential equation of beam bending to account for hinge compliance. To describe angular displacements at the interfaces between beam segments, the delta-function method and the Heaviside function are employed, which enables the incorporation of local effects into the computational model.

The contact layer method has proven to be an effective tool for accounting for adhesive interactions between the layers of multilayer beams. This method is based on representing the contact layer as a transversely anisotropic medium composed of short, non-interconnected elastic rods. Such a formulation avoids inaccuracies inherent in classical models, such as the occurrence of infinite shear stresses at interlayer boundaries. An improved technique, combining the finite element method with the contact layer method, extends the scope of application and enables the modeling of layered beams in MATLAB, taking into account transverse shear deformations and variable boundary conditions of the layers. These methods provide higher modeling accuracy and enables assessment of the influence of contact stiffness and shear deformations on the stress-strain state of the structure [11], [12].

In [13], a simplified finite element model was presented for analyzing a beam on a two-parameter elastic foundation. The model incorporates foundation characteristics (stiffness and shear resistance), formulates stiffness and reaction force matrices, and performs numerical calculations under various boundary conditions to validate its applicability. The results were compared with those of the well-known models by Winkler and Vlasov, confirming the applicability of the proposed approach in engineering practice. Contemporary research is focused on developing universal models capable of accounting not only for vertical but also for shear stresses, multilayer beam configurations, and variable stiffness of intermediate layers. This research direction is particularly relevant for the analysis of composite structures, where different materials jointly sustain complex loading conditions. Increasing attention has been given to the use of numerical methods, including the finite element method, which enables the incorporation of complex geometries, boundary conditions, and nonlinear effects such as plasticity, cracking, and creep in materials [14], [15]. Models considering geometric and physical nonlinearities are crucial for stress analysis under real operating conditions. Advances in finite and boundary element methods, as well as variational approaches, have improved the accuracy of modeling structures on deformable foundations. Recent studies emphasize achieving both analytical interpretability and computational efficiency, while also ensuring experimental verification to enhance practical applicability in engineering [16], [17].

The relevance of this research arises from the identified research gap associated with the limitations of classical models for analyzing structures on elastic foundations. In particular, Winkler's model, which represents the foundation as a system of independent elastic elements, fails to capture the actual stress-strain state of the soil. Vlasov's model, although extending the formulation of the problem, also introduces simplifications, notably neglecting lateral interactions that significantly affect structural behavior. The absence of a comprehensive model in existing publications justifies the need for further research. The identified research gap demonstrates that there is still no comprehensive model that adequately captures both normal and shear stresses in beams on elastic foundations while maintaining computational efficiency. Therefore, the aim of this study is to develop an advanced mathematical model of a two-parameter elastic foundation, which eliminates the limitations of existing approaches, provides a more accurate description of the stress-strain state, and enhances the reliability of engineering design.

2. Methods

Consider a straight elastic beam of length l and thickness h_0 composed of a homogeneous isotropic material with elasticity modulus E. The beam is defined in a Cartesian coordinate system as $\left(-\frac{l}{2} \leq x_1 \leq \frac{l}{2}, -\frac{h_0}{2} \leq x_3 \leq \frac{h_0}{2}\right)$. The foundation is modeled as a 2-parameter elastic medium of finite thickness h, with its material characterized by an effective modulus of elasticity $\bar{E}$ (Figure 1) [18].

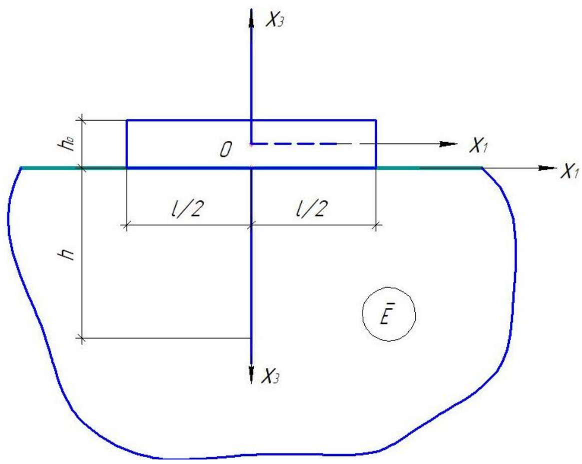

Figure 1 – Beam on a two-parameter elastic foundation

In the problem of beam-foundation interaction, where the foundation is modeled as a linearly deformable elastic half-plane, a key step is the correct formulation of the interface conditions between the structural elements. These conditions ensure the continuity of displacements and the consistency of the stress state at the beam-foundation interface. The coupling condition can be formulated as follows: the deflection of the beam must coincide with the deflection of the elastic foundation surface, i.e., $W_0(x_1) = W(x_1)$.

For the modeling of the elastic foundation in stress-strain analysis, the displacement function method is employed. Considering the effective modulus of elasticity $\bar{E}$ and the shear modulus G of the foundation material, the governing equilibrium equation can be expressed as follows:

$$\nabla^2\nabla^2 F = \frac{\partial^4 F}{\partial x_1^4} + 2\frac{\partial^4 F}{\partial x_1^2 \partial x_3^2} + \frac{\partial^4 F}{\partial x_3^4} = 0 \tag{1}$$

To solve the biharmonic Eq. (1), the displacement $F(x_1, x_3)$ is defined in the following form [19]:

$$F(x_1, x_3) = \delta(x_3) \cdot W(x_1), \tag{2}$$

where: $\delta(x_3)$ is the distribution function, and $W(x_1)$ is the foundation deflection function.

The governing Eq. (1), taking into account the transition relations $\frac{d^2W(x_1)}{dx_1^2} = -\bar{k}^2 W(x_1)$, $\frac{d^4W(x_1)}{dx_1^4} = \bar{k}_\omega^4 W(x_1)$ and the prescribed form of the displacement Eq. (2), can be written as follows:

$$\delta^{IV}(z_0) - 2 \cdot k^2 \delta''(z_0) + k_\omega^4 \delta(z_0) = 0 \tag{3}$$

The general solution of this equation is given by the expression $\lambda^2 = k^2\left(1 \pm \sqrt{1-\alpha}\right)$; $k_\omega^4 = \alpha \cdot (k^2)^2$:

$$1.\ \alpha = 1:\ \ \delta(z_0) = (C_1 + C_2 z_0)e^{-kz_0} + (C_3 + C_4 z_0)e^{kz_0} \tag{4}$$

$$2.\ \alpha > 1:\ \ \delta(z_0) = [C_1 \cos(\beta_1 z_0) + C_2 \sin(\beta_1 z_0)]e^{-\alpha_1 z_0} + [C_3 \cos(\beta_1 z_0) + C_4 \sin(\beta_1 z_0)]e^{\alpha_1 z_0} \tag{5}$$

$$3.\ \alpha < 1:\ \ \delta(z_0) = C_1 e^{\alpha_2 z_0} + C_2 e^{-\alpha_2 z_0} + C_3 e^{\beta_2 z_0} + C_4 e^{-\beta_2 z_0}, \tag{6}$$

where: $\alpha_1 = k\sqrt{\frac{\alpha+1}{2}}$; $\beta_1 = k\sqrt{\frac{\alpha-1}{2}}$; $\alpha_2 = k\sqrt{1-\sqrt{1-\alpha}}$; $\beta_2 = k\sqrt{1+\sqrt{1-\alpha}}$.

The results obtained from Eq. (4) are discussed in detail in [19]. In the present study, the main focus is placed on the analysis of Eq. (5) and the derivation of the corresponding solutions. The general solution of Eq. (5) is applied to the problem of a semi-infinite elastic half-plane, which serves as the basis for the subsequent analysis ($z_0 \to \infty, \delta(z_0) = 0$):

$$\delta(z_0) = [C_1 cos(\beta_1 z_0) + C_2 \sin(\beta_1 z_0)]e^{-\alpha_1 z_0} \tag{7}$$

When applying the displacement method, the unknown constants, parameters, and the governing equation included in the fundamental relations are determined as follows. The unknown constants:

$$C_1 = \frac{1}{12\alpha_p}[2\nu\alpha_1\alpha\alpha_0 + (\alpha - \nu)\beta_0]$$

$$C_2 = -\frac{1}{12\beta_1\alpha_p}[(\alpha + \nu)\alpha_1 \cdot \beta_0 - (1+\nu)\alpha k^2 \alpha_0] \tag{8}$$

$$C_0 = \frac{1}{12}\frac{\bar{E}h^2}{Eh_0^2}\beta_0;\ \ A_0 = -\frac{1}{8} + \frac{1}{24}\frac{\bar{E}h^2}{Eh_0^2}\beta_0;\ \mathrm{B}_0 = \frac{1}{24} - \frac{1}{32}\frac{\bar{E}h^2}{Eh_0^2}\beta_0 - \frac{1}{12}\frac{\bar{E}h^3}{Eh_0^3}\alpha_0$$

The parameters of the computational model are defined in terms of Poisson’s ratio ν, the stiffness characteristics of the foundation, and the relationships between its geometric dimensions and those of the beam. These parameters provide the basis for constructing the model and conducting further analytical investigations.

$$\alpha_0 = \frac{6(1-\nu)}{\alpha k^2(1+\nu)}\alpha_p P_1;\ \beta_0 = 12(1-\nu)\alpha_p P_0;$$

$$P_0 = \frac{1 - \alpha\nu + \alpha_1\frac{h_0}{h}}{(1-\alpha\nu)n_1k^2 - 2\alpha_1 m};\ P_1 = \frac{2m + n_1k^2\frac{h_0}{h}}{(1-\alpha\nu)n_1k^2 - 2\alpha_1 m} \tag{9}$$

$$n_1 = \alpha^2(1-\nu) - \alpha\nu(1+\nu) + 2\nu;\ m = 2\alpha\alpha_1(1+\nu) + \alpha_p(1-\nu)\frac{\bar{E}h}{Eh_0};$$
$$\alpha_p = \alpha^2\nu + 2\alpha\nu + \nu^2\alpha + \alpha - \nu$$

Based on Eqs. (8) and (9), the governing equation is formulated, which defines the stress-strain state of a beam resting on an elastic foundation:

$$\gamma \cdot \frac{d^4W_0(x_1)}{dx_1^4} = \frac{q(x_1)}{EJ};\ J = \frac{h_0^3}{12};\ \gamma = 1 - 6(1-\nu)\alpha_p \cdot P_0\frac{\bar{E}h^2}{Eh_0^2} - \frac{6(1-\nu)\alpha_p \cdot P_1}{\alpha k^2(1+\nu)}\frac{\bar{E}h^3}{Eh_0^3} \quad (10)$$

Taking into account the transition relations, the derived governing Eq. (10) can be reduced to the form of the standard beam-on-elastic-foundation equation, which has been widely applied in numerous theoretical and applied studies: $\frac{d^4W_0(x_1)}{dx_1^4} - 2r^2\frac{d^2W_0(x_1)}{dx_1^2} + S^4 \cdot W_0(x_1) = \frac{q(x_1)}{EJ}$; $2r^2 = -\frac{6(1-\nu)\alpha_p k^2 \cdot P_0}{h^2}\frac{\bar{E}h^2}{Eh_0^2}$; $S^4 = -\frac{6(1-\nu)\alpha_p k^2 \cdot P_1}{(1+\nu)h^4}\frac{\bar{E}h^3}{Eh_0^3}$.

The bending moment M and the shear force Q of the beam are determined through integral relations that incorporate the normal and shear stresses along the section height. As a result, expressions are obtained that establish the relationship between these internal force factors and the deflection function W_0, the flexural rigidity EJ, as well as the foundation parameters g_1 and g_2, which are defined by integrals of the functions $\varphi_0(z)$ and $\psi_0(z)$ [20].

$$M = h_0^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\sigma_1^0 \cdot zdz = -EJ \cdot g_1 \cdot \frac{d^2W_0(x_1)}{dx_1^2}$$
$$Q = h_0\int_{-\frac{1}{2}}^{\frac{1}{2}}\tau_{13}^0 \cdot dz = EJ \cdot g_2 \cdot \frac{d^3W_0(x_1)}{dx_1^3} \quad (11)$$
$$g_1 = 12\int_{-1/2}^{1/2}\varphi_0(z) \cdot zdz, \quad g_2 = 12\int_{-1/2}^{1/2}\psi_0(z)dz$$

The normal and shear stresses in the beam are defined through the internal forces according to Eq. (11). These relations establish a link between the stress state of the material and the integral parameters characterizing the section behavior, namely the bending moment and the shear force. This procedure enables a more detailed analysis of the stress distribution along the section height and accounts for the influence of the geometric and mechanical parameters of the beam on its stress-strain state [21].

$$\sigma_1^0 = h_0 \cdot \varphi_0(z)\frac{M}{J}$$
$$\tau_{13}^0 = -h_0^2 \cdot \psi_0(z)\frac{Q}{g_0 J} \quad (12)$$
$$\sigma_3^0 = h_0^3 \cdot \delta_0(z)\frac{q(x_1)}{\gamma \cdot J}$$
$$\psi_0(z) = A_0 - C_0 z + \frac{z^2}{2}$$
$$\delta_0(z) = B_0 - A_0 \cdot z + C_0\frac{z^2}{2} - \frac{z^3}{6},$$

where: $\psi_0(z)$ and $\delta_0(z)$ are the stress distribution functions.

The displacement components of the beam are obtained by integrating the expressions describing linear and transverse shear deformations [22]. This calculation scheme establishes an analytical relationship between the deflection function and the internal deformation parameters, thereby enabling a consistent consideration of shear effects in the development of the computational model.

$$U_3^0(x_1, x_3) = W_0(x_1)$$
$$U_1^0(x_1, x_3) = -h_0 \cdot \varphi_0(z)\frac{dW_0(x_1)}{dx_1} \quad (13)$$
$$\varphi_0(z) = -C_0 + z; \quad z = \frac{x_3}{h_0},$$

where: $\varphi_0(z)$ is the shear displacement distribution function (U_1^0); $W_0(x_1)$ is the beam deflection function (normal displacement); C_0 is an undetermined constant; z is the dimensionless transverse coordinate.

At the beam ends, one of the following boundary conditions must be satisfied [23]:

1) In the case where the beam end is in full contact with the elastic foundation:

$$W_* = \frac{Q \cdot L^3}{3\bar{E}J_0};\ \varphi_* = \frac{M \cdot L}{\bar{E}J_0};\ J_0 = \frac{h^3}{12}, \tag{14}$$

where: W_* and φ_* are vertical and angular displacements at the beam end; Q, M are the internal forces (shear force and bending moment, respectively) at the corresponding beam ends; L is the length of the deformable foundation; $\bar{E}J_0$ is the bending stiffness of the deformable foundation (h is its thickness; $\bar{E}$ is the elastic modulus of the foundation material);

2) In the case where a hinged support is installed at the beam end:

$$W_0 = 0; \quad M = -EJ\frac{d^2W_0(x_1)}{dx_1^2} = 0 \tag{15}$$

3) In the case where the beam end is rigidly clamped:

$$W_0 = 0; \quad \theta = \frac{dW_0(x_1)}{d(x_1)} = 0 \tag{16}$$

where: θ is the rotation angle.

4) In the case where the beam end is free (no contact) [23]:

$$M = -EJ\frac{d^2W_0(x_1)}{dx_1^2} = 0;\ Q = -EJ\mathrm{g}_0\frac{d^3W_0(x_1)}{dx_1^3} = 0 \tag{17}$$

For a beam resting on a two-parameter elastic foundation, the construction of the analytical model requires consideration of the following fundamental parameters:

1) $\mathrm{E}, h_0, l, q(x_1)$ – modulus of elasticity of the beam material, beam thickness (height), beam length, and the magnitude of the uniformly distributed load;

2) $\bar{E}, \nu, h, L$ – modulus of elasticity of the elastic foundation material, Poisson's ratio, thickness, and the extent of the region over which the beam's influence spreads beyond the contact zone.

To determine the stress-strain state of the beam-foundation interaction, the following sequence of computational steps is performed:

1) The governing Eq. (10) is solved subject to boundary conditions (14)-(17), yielding the beam deflection function $W_0(x_1)$;

2) Internal forces are computed according to Eq. (11);

3) Stress components $\sigma_1^0, \tau_{13}^0, \sigma_3^0$ are determined using Eq. (12).

3. Results and Discussion

Analytical solutions have been derived for various boundary conditions, loading types, and for cases involving variations in the geometric and physical-mechanical characteristics of both the beam and the foundation. To ensure the reliability of the analysis and enable a meaningful comparison of results, numerical simulations were carried out in the Mathcad environment, providing a basis for validating the analytical solutions. A beam resting on a two-parameter elastic foundation is considered. The applied load is assumed to be a uniformly distributed load of $q=1\ kN/m$. The beam length is taken as $l=1\ m$ and the height is $h_0=0.25m$. The beam is characterized by an elastic modulus of $E=1\cdot 10^{10}Pa$. The physical and geometry parameters of the elastic foundations were $\bar{E} = 1Pa$, $\nu = 0.25$ and $h = 1m$.

The results of the comparison between the classical and the proposed methods for analyzing a beam on an elastic foundation, including deflections, bending moments, and shear forces, are presented in Tables 1-3 and Figures 2-4. The analysis demonstrates that the values obtained using the proposed method are in good agreement with those derived from the Winkler model.

Table 1 – Vertical displacement values

Case	The length of the beam				
	0	0.25	0.5	0.75	1.0
y	0	0.000711904	0.000999157	0.000711904	0
Wp	0	0.00071119	0.00099816	0.00071119	0

Table 2 – Bending moment values

Case	The length of the beam				
	0	0.25	0.5	0.75	1.0
Mv	0	0.093673	0.124892	0.093673	0
Mp	0	0.09358	0.12477	0.09358	0

Table 3 – Shear force values

Case	The length of the beam				
	0	0.25	0.5	0.75	1.0
Qv	0.49966	0.24976	0	-0.24976	-0.49966
Qp	0.49892	0.24946	0	-0.24946	-0.49892

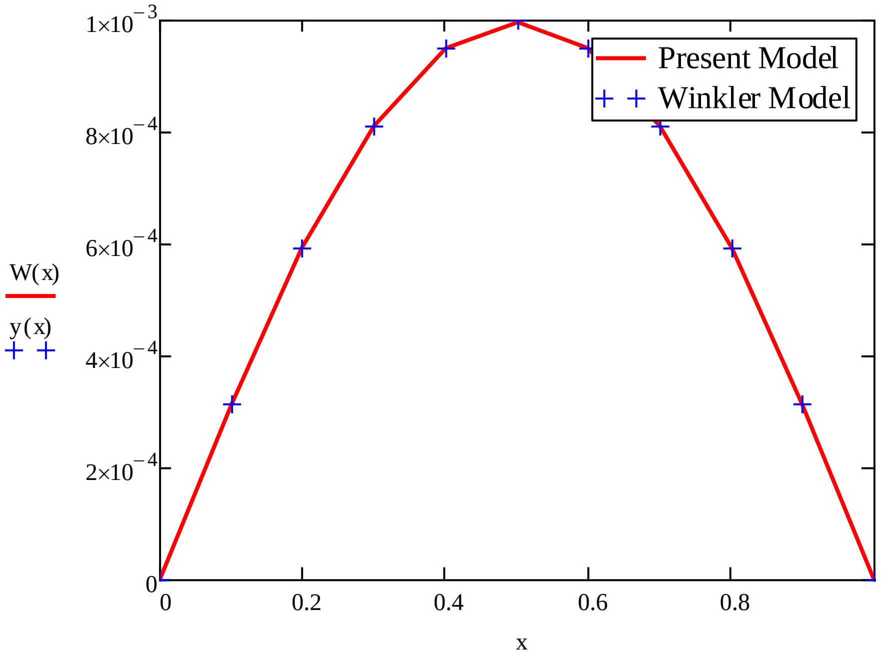

Figure 2 – Vertical displacement of the beam on a two-parameter elastic foundation

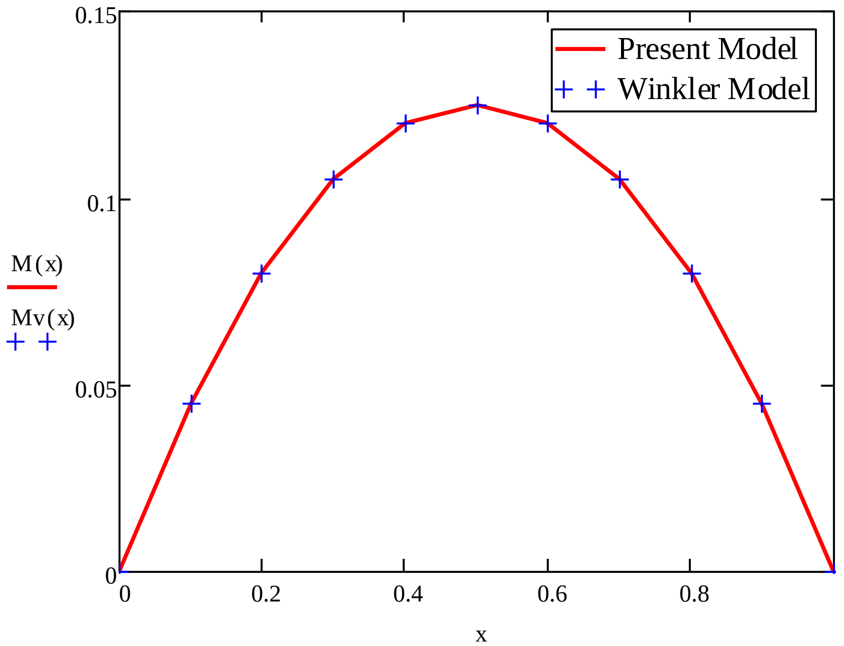

Figure 3 – Bending moment of the beam on a two-parameter elastic foundation

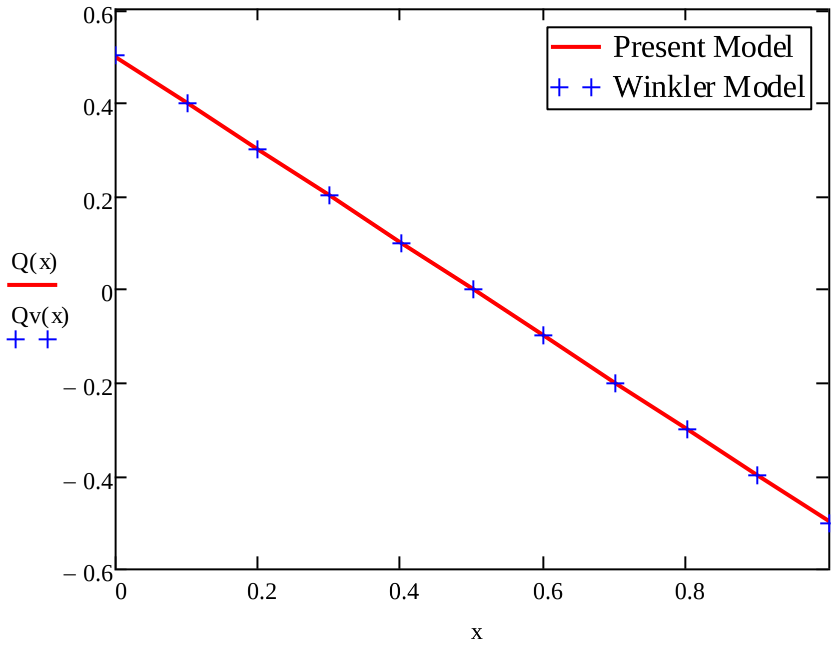

Figure 4 – Shear force of the beam on a two-parameter elastic foundation

The graphical results (Figures 2-4) for vertical displacement, bending moment, and shear force showed close agreement between the present study, the simplified formulation, and Winkler's classical solution. Tables 1, 2, and 3, as well as Figures 2, 3, and 4, present the calculation results for different beam lengths. As can be seen from the tables, the values obtained by the Present method showed good agreement with those calculated using the Winkler model. Maximum deflection was observed at the mid-span, while displacements decreased symmetrically toward the supports, and the distributions of bending moments and shear forces corresponded to theoretical expectations. The close coincidence of the curves confirmed the validity and reliability of the developed method, while the two-parameter model remained consistent with the Winkler approach and simultaneously provided broader applicability for more complex loading and heterogeneous foundation conditions.

Tables 4 and 5 present the values of shear and normal stresses calculated using the proposed method τ_{130}, σ_{10} and the Winkler model τ_{13v}, σ_{1v} at different points of the beam cross-section. The analysis shows that the maximum shear stresses are observed near the fixed end of the beam (x=0, z=0), gradually decrease to zero at midspan, and reach negative values at the opposite boundary (x=1, z=1). The normal stresses vary linearly along the cross-sectional height: from negative in the lower zone (z=−1/2) to positive in the upper zone (z=1/2), while their values are close to zero in the region of the neutral axis (z=0). The comparative analysis indicates a high degree of consistency between the results of the two methods, as evidenced by the nearly coincident values. Figure 5 shows the distribution of shear stresses along the beam cross-sectional height, while Figure 6 illustrates the distribution of normal stresses. In both cases, a high degree of agreement is observed between the results obtained by the proposed method and those of the classical Winkler model.

Table 4 – Shear stress results

Case	x=0, z=0	$x=\frac{1}{4}$, z=0	$x=\frac{1}{2}$, z=0	$x=\frac{3}{4}$, z=0	x=1, z=0
τ_{130}	2.995	-1,498	0	-1,498	-2,995
τ_{13v}	2.998	-1,499	0	-1,499	-2,998

Table 5 – Normal stress results

Case	$x=\frac{1}{2}$, $z=-\frac{1}{2}$	$x=\frac{1}{2}$, $z=-\frac{1}{4}$	$x=\frac{1}{2}$, z=0	$x=\frac{1}{2}$, $z=\frac{1}{4}$	$x=\frac{1}{2}$, $z=\frac{1}{2}$
σ_{10}	-11.977	-5.988	0.001217	5.988	11.977
σ_{1v}	-11.99	-5.995	0	5.995	11.99

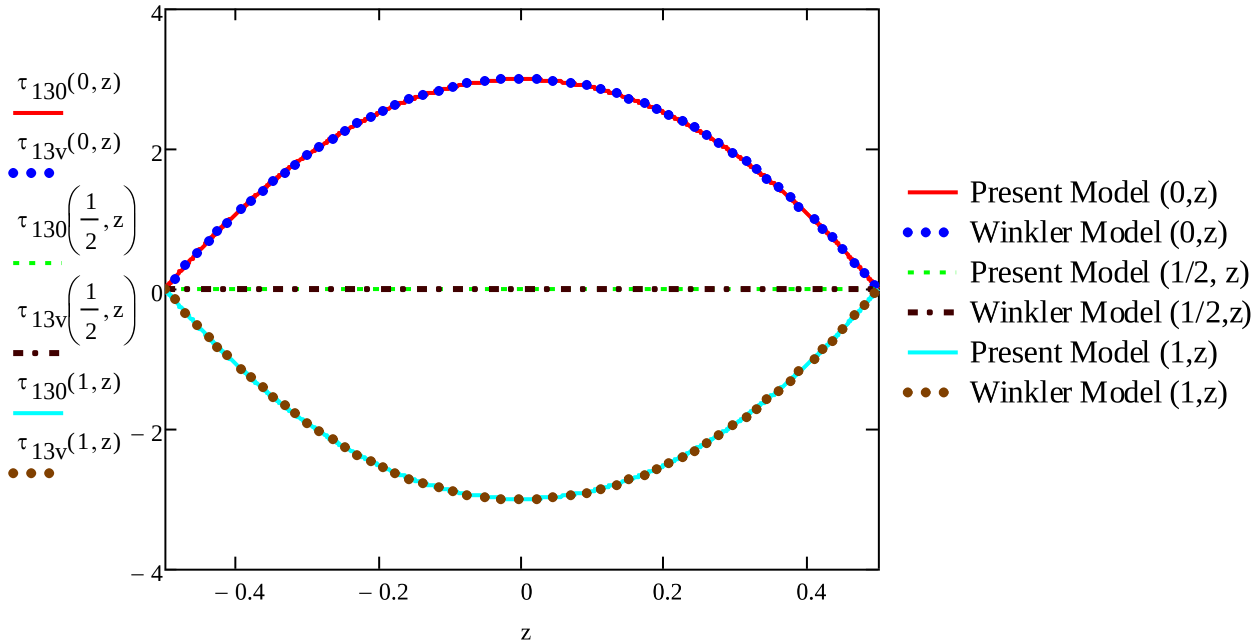

Figure 5 – Distribution of shear stresses

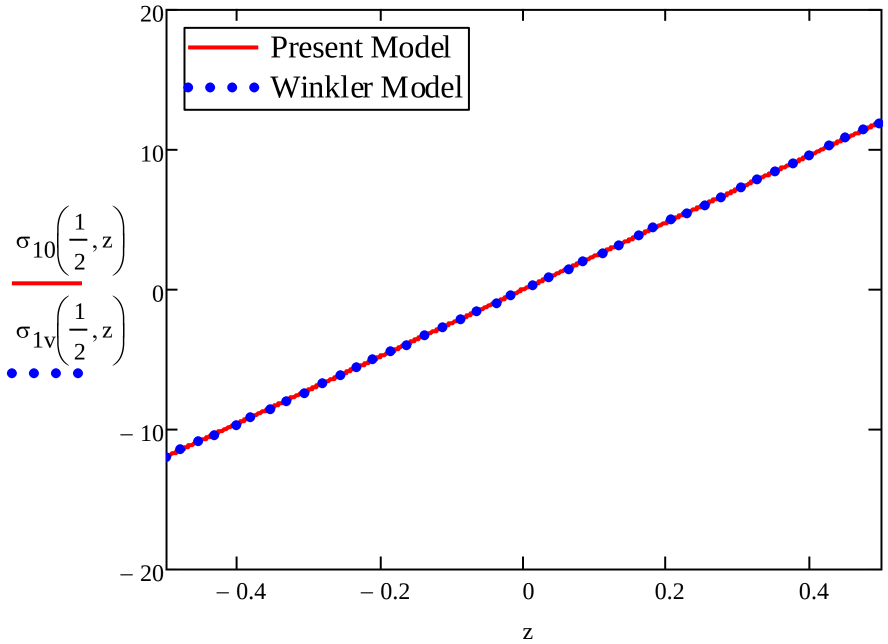

Figure 6 – Normal stress distribution

Table 4 and Figure 5 present the shear stress values at the point $z = 0$ for varying coordinate x across five calculation points. As the results show, the values obtained using the proposed method are in good agreement with those calculated based on the Winkler model. Table 5 and Figure 6 provide the normal stress values at a constant x with varying coordinate z. These results also demonstrate consistency with the Winkler model, thereby confirming the validity of the proposed method. The comparative analysis demonstrated a high degree of consistency between the calculated stresses, confirming the reliability and applicability of the considered methods in engineering practice. The obtained findings not only enhanced the understanding of beam behavior on elastic foundations but also provided a basis for developing practical recommendations aimed at improving the accuracy and reliability of structural design.

4. Conclusions

An analytical solution has been developed for the stress analysis of beams resting on a two-parameter elastic foundation. The proposed mathematical model accounts for both normal and shear stresses, as well as the influence of the elastic foundation, enabling calculations for various boundary conditions and loading scenarios. The accuracy of the method has been verified through analytical examples and numerical simulations, including comparisons of vertical displacements, bending moments, and shear forces. Comparative evaluation with the classical Winkler model shows good agreement for vertical displacements, bending moments, and shear forces, while the proposed method provides improved accuracy in predicting both shear and normal stress distributions. The results confirm that the refined approach is reliable, practically applicable, and capable of capturing detailed stress-strain behavior that classical models may overlook. Overall, the developed simplified theory of a beam on a two-parameter elastic foundation is of significant interest to structural engineers involved in foundation design. It not only aligns well with classical Winkler predictions but also enhances the precision of stress analysis, providing a robust basis for practical applications and serving as a foundation for further research in structural mechanics and geotechnics.

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